5. Khaskind, M. D., Unsteady motion of a rigid body in an accelerating flow of an infinite fluid. PMM Vol. 20, №1, 1956.
6. Gurevich, M. I., Aerodynamic effect of a train on a small body. Izv. Akad. Nauk SSSR, MZhG, №3, 1968.
7. Iakimov, Iu. L., The motion of a cylinder in an arbitrary plane flow of a perfect incompressible fluid. Izv. Akad. Nauk SSSR, MZhG, №2, 1970.
8. Kostiukov, A. A., Interaction between bodies moving in a fluid. Sudostroenie, Leningrad, 1972.
9. Sedov, L. I., Mechanics of Continuous Medium. Vol. 2, "Nauka", Moscow, 1970.
10. Milne-Thomson, L. M., Theoretical Hydrodynamics, 6th ed. Macmillan, 1968.
11. Lamb, H., Hydrodynamics. Cambridge University Press, 1953.
12. Serrin, J., Mathematical Foundations of Classical Mechanics of Fluids. Izd. Inostr. Lit., Moscow, 1963.
13. Appel, P. E., Traité de Mécanique Rationelle, Vol. 2, Dynamique des Systèmes, 1893.
14. Kochin, N. E., Kibel, I. A. and Roze, N. V., Theoretical Hydrodynamics, pt. 1. Fizmatgiz, Moscow, 1963.

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ON THE ASYMPTOTIC BEHAVIOR OF A STEADY FLOW OF VISCOUS FLUID AT SOME DISTANCE FROM AN IMMERSED BODY

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The steady flow of a viscous incompressible fluid past a body of finite dimensions is considered. It is assumed that the velocity vector u satisfies condition

$$u - u_\infty = O(R^{-\alpha})$$

where u_∞ is the velocity vector of the oncoming stream, R is the distance from a fixed point of the body, and $\alpha > 1/2$. Terms defining the asymptotic behavior of velocity of the order of $O(R^{-1})$ and $O(R^{-3/2})$ are determined and an estimate of the residual term is given. The derived asymptotic formula for the velocity vortex shows that outside the wake the vortex decreases according to an exponential law.

1. Lemmas. 1.1. Let us consider the steady flow of a viscous incompressible fluid past a body such that $B \subset R^3$. We denote the dimensionless velocity vector and pressure by u and p, respectively. Let $S = \partial B$ be a surface which satisfies the Liapunov conditions. We locate the coordinate origin inside B and select the direction of coordinate axes and the scale so that the oncoming stream velocity u_∞ is (1, 0, 0) and the diameter B is unity.

The steady motion of a viscous fluid is defined by the system of equations

$$u \cdot \nabla u + \text{grad } p = \Delta u / 2\lambda, \qquad \text{div } u = 0 \tag{1.1}$$

where 2λ denotes the Reynolds number. Let us define boundary conditions at the body by

$$u|_S = u_0 \tag{1.2}$$

where function u_0 is subject to condition

$$\int_S u_0 \cdot n \, d\sigma = 0 \tag{1.3}$$

in which n is the unit vector of the inward normal to surface S and $d\sigma$ is the Lebesgue measure on S. At infinity the condition

$$\lim_{|x|\to\infty} u(x) = u_\infty \tag{1.4}$$

must also be satisfied.

The existence of solution of the problem defined by (1.1), (1.2) and (1.4), when condition (1.3) is satisfied, was first proved in [1, 2], then in [3, 4], in [5] and in [6]. In papers [1, 2] the condition (1.4) is satisfied in some general sense and, if the fluid is at rest at infinity, also in the classical sense. All authors of the above cited works had established the existence of solution in the class of solutions containing the finite Dirichlet integral

$$\int_G |\nabla u|^2 \, dx < \infty \tag{1.5}$$

where $G = R^3 \setminus B$. Finn had shown [7] that (1.4) can be satisfied in any solution of flow subjected to condition (1.5), and Faddeev had proved this for the class of generalized solutions derived by Ladyzhenskaia.

The problem of asymptotic behavior of solutions at some distance from the body is of fundamental interest, if only in relation to the boundary layer theory. A refinement of formula (1.4), at least with respect to the order of magnitude of the decrease of $u(x) - u_\infty$, proved to be unsuccessful up to the present.

The series of detailed investigations by Finn and his collaborators of the asymptotics of this solution, made on the assumption that

$$u(x) - u_\infty = O(|x|^{-\alpha}) \tag{1.6}$$

is possibly related to this aspect. Finn established that for $\alpha > 1/2$ the asymptotics of the velocity vector is defined by

$$u(x) = u_\infty + H(x) \cdot a + O(|x|^{-3/2+\delta}) \tag{1.7}$$

where a is the vector of force exerted by the stream on the body, $H(x)$ is the Green's matrix of the Oseen system, and δ is an arbitrarily small positive number. The asymptotics of derivatives $\partial u / \partial x_k$ is obtained by formal differentiation of formula (1.7) which trivially yields formula (1.5).

It was shown in a recent paper by Babenko (*) that formula (1.6) in which α can be made arbitrarily close to unity is satisfied for any solution of the problem of flow with a finite Dirichlet integral. The asymptotic formula (1.7) and its refinements are, consequently, valid for solutions with a finite Dirichlet integral.

A refinement of formula (1.7) is presented here and the rate of vorticity decrease at

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*) K. I. Babenko, "On stationary solutions of the problem of viscous incompressible fluid flow past a body", preprint by the Institute of Applied Mathematics, AN SSSR. Deposited №4815-72.

some distance from the body is investigated. Almost all results obtained here were indicated in the preprint of a paper by the authors (*). The results of independent investigations of vorticity decrease appear in [8]. An estimate of vorticity decrease, when a certain one-sided inequality is satisfied, was obtained in [9]. It should be noted that the method developed in [10] for estimating the vorticity decrease is equally suitable for investigations of plane and three-dimensional flows. This method is used below.

1.2. Let us set $u = v + u_\infty$ and consider the system of Oseen's equations

$$\Delta v - 2\lambda \frac{\partial v}{\partial x_1} - 2\lambda \operatorname{grad} p = 0, \quad \operatorname{div} v = 0$$

The fundamental solution of this system is of the form

$$H_{ij}(x-y) = \delta_{ij}\Delta\Phi - \frac{\partial^2\Phi}{\partial x_i \partial x_j}, \quad i, j = 1, 2, 3$$

$$2\lambda q_i(x-y) = -\frac{\partial}{\partial x_i}\left(\Delta\Phi - 2\lambda \frac{\partial \Phi}{\partial x_1}\right), \quad i = 1, 2, 3.$$

where

$$\Phi = \Phi(s) = -\frac{1}{8\pi\lambda}\int_0^{\lambda s}(1 - e^{-t})\frac{dt}{t}$$

$$s = |x-y| - x_1 + y_1, \quad x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3)$$

It can be shown that the estimates

$$\left|\frac{\partial^l}{\partial x_k^{\,l}} H_{ij}(x-y)\right| \leqslant C\,[|x-y|^{-1-l/2}(s+1)^{-1-l/2} + |x-y|^{-1-l}(s+1)^{-1}] \tag{1.8}$$

where $l = 0, 1$ are valid for the fundamental solution. It can be readily verified that

$$\left|\frac{\partial}{\partial x_1} H_{ij}(x-y)\right| \leqslant C|x-y|^{-2}(s+1)^{-1}, \qquad |q_i(x-y)| \leqslant C|x-y|^{-2} \tag{1.9}$$

We point out that the letter C, whether with or without a subscript denotes here various constants which depend only on λ.

1.3. Let us denote by F the vector whose coordinates are

$$F_i = 2\lambda \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k}, \qquad i = 1, 2, 3$$

Finn [7] has shown that, when (1.5) is satisfied, $v(x)$ can be represented by Green's formula

$$v(x) = I_0(x) + \int_G H(x-y)F(y)\,dy \tag{1.10}$$

where H is a matrix with elements H_{ij}, and $I_0(x)$ is a vector whose components are

$$I_{0k}(x) = \int_S \Big[\sum_{j,l=1}^{3}\left(\frac{\partial H_{kl}}{\partial y_j} + \frac{\partial H_{kj}}{\partial y_l} + 2\lambda\delta_{jl}q_k\right)v_l n_j + \lambda n_1 \sum_{l=1}^{3} H_{kl}v_l\Big]d\sigma - \int_S \sum_{j,l=1}^{3} H_{kl}\left(\frac{\partial v_l}{\partial y_j} + \frac{\partial v_j}{\partial y_l} - 2\lambda\delta_{jl}p\right)n_j d\sigma, \quad k = 1, 2, 3 \tag{1.11}$$

*) K. I. Babenko and M. M. Vasil'ev, "The asymptotic behavior of solutions of the problem of viscous fluid flow past a finite body", preprint by the Institute of Applied Mathematics, AN SSSR, Deposited №4590-72.

Denoting the volume integral in (1.10) by $I_d\ (x)$, after integration by parts, for the components of vector I_d we obtain

$$I_{dk} = -2\lambda \int_G \sum_{j,\,l=1}^{3} \frac{\partial H_{kl}}{\partial y_j} v_l v_j\, dy + 2\lambda \int_S \sum_{j,\,l=1} H_{kl} v_j v_l n_j\, d\sigma \qquad (1.12)$$

Let us set

$$J_{dk}(x) = -2\lambda \int_G \sum_{j,\,l=1}^{3} \frac{\partial H_{kl}}{\partial y_j} v_l v_j\, dy$$

combine the surface integral in formula (1.12) with the integral (1.11), and denote the result by J_{0k}. Thus

$$v\ (x) = J_0\ (x) + J_d\ (x) \qquad (1.13)$$

2. Derivation of principal terms of asymptotics. 2.1. The derivation of principal terms of the asymptotics is based on the evaluation of integrals (1.13). To do this we introduce certain lemmas. Let us consider the convolution

$$I(x) = \int_{R^3} W(x-y) f(y)\, dy \qquad (2.1)$$

and assume that

$$|f(x)| \leqslant (|x|+1)^{-\beta}(s+1)^{-\gamma},\ |W(x)| \leqslant |x|^{-\delta}(s+1)^{-\varepsilon}$$

where β, γ, δ and ε are nonnegative constants and $\varepsilon \geqslant 1$. Let us estimate $I\ (x)$ for large $|x|$.

We set $|x| = R$, $\theta |x| = R_0$, where $\theta = \text{const}$ and $0 < \theta \leqslant 1/4$. Let

$$D_0 = \{y\colon |y_1| \leqslant R_0,\ y_2^2 + y_3^2 \leqslant R_0^2\}$$

$$D_x = \{y\colon |y_1 - x_1| \leqslant R_0,\ (y_2-x_2)^2 + (y_3-x_3)^2 \leqslant R_0^2\}$$

On these assumptions $D_0 \cap D_x = \phi$. Let us set $D = R^3 \setminus D_0 \cup D_x$. In conformity with the subdivision of R^3 we represent $I\ (x)$ as the sum of three integrals taken over regions D_0, D_x and D and denote these by I_1, I_2 and I_3, respectively. If the parts played by functions f and W are interchanged, then, by the substitution of y for $x-y$, integral I_2 reduces to integral I_1. It is, however, more convenient to consider I_2 separately, since function W must satisfy the condition $\varepsilon \geqslant 1$, while no conditions whatsoever are imposed on γ.

Let us estimate integral I_3. It can be shown that

$$C^{-1} \leqslant \frac{|y-x|}{|y|} \leqslant C, \qquad \forall y \in D$$

We verify that

$$[s\,(y)+1]\ [s\,(x-y)+1] \geqslant s\,(x)+1$$

Hence, setting $\omega_h = \min\,[\gamma,\ \varepsilon,\ \max\,(\gamma,\ \varepsilon) - 1 - h]$, where $|h| \leqslant 1$ ($h = \text{const}$), we obtain

$$|I_3(x)| \leqslant C\,[s(x)+1]^{-\omega_h} \int_{|y|\geqslant R_0} |y|^{-\beta-\delta}\,[s(y)+1]^{-1-h}\, dy$$

If $h_0 = \max\,(-h,\ 0)$ is chosen such that $\beta + \delta > 2 + h_0$, then

$$\int_{|y|\geqslant R_0} |y|^{-\beta-\delta}\,[s(y)+1]^{-1-h}\, dy = 2\pi \int_{R_0}^{\infty} \rho^{2-\beta-\delta}\, d\rho \int_0^{\pi} \frac{\sin\varphi\, d\varphi}{[\rho\,(1-\cos\varphi)+1]^{1+h}} \leqslant CR_0^{2-\beta-\delta+h_0}\Delta_{0,\,h}$$

Here and subsequently we use the notation

$$\Delta_{a,b} = \begin{cases} 1, & \text{if } a \neq b \\ \log R, & \text{if } a = b \end{cases}$$

It follows from the last estimates that

$$|I_3(x)| \leqslant CR^{2-\beta-\delta+h_0} [s(x)+1]^{-\omega_h} \Delta_{0,h} \qquad (2,2)$$

2.2. Let us consider the integral $I_2(x)$ for $\zeta_h = \min(\gamma, \varepsilon - 1 - h)$. We have

$$|I_2(x)| \leqslant C [s(x)+1]^{-\zeta_h} |x|^{-\beta} \int\limits_{|y|\leqslant R_0} |y|^{-\delta} [s(y)+1]^{-1-h} dy$$

for $\delta < 2 + h_0$ the estimate of the last integral is

$$|I_2(x)| \leqslant CR^{2-\beta-\delta+h_0} [s(x)+1]^{-\zeta_h} \Delta_{0,h} \qquad (2,3)$$

2.3. Let us estimate integral $I_1(x)$. Let $r = \sqrt{x_2^2 + x_3^2}$. We subdivaide D_0 into three regions

$$d_1 = \left\{y : |y_1| \leqslant R_0,\ y_2^2 + y_3^2 \leqslant \left(\frac{\theta r}{4}\right)^2\right\}$$

$$d_2 = \left\{y : |y_1| \leqslant R_0,\ (y_2 - x_2)^2 + (y_3 - x_3)^2 \leqslant \left(\frac{\theta r}{4}\right)^2\right\}$$

$$d_3 = D_0 \setminus d_1 \cup d_2$$

In conformity with this subdivision we represent $I_1(x)$ as the sum of three integrals

$$I_1(x) = I_{11}(x) + I_{12}(x) + I_{13}(x) \qquad (2.4)$$

where I_{11}, I_{12} and I_{13} are integrals taken over regions d_1, d_2 and d_3, respectively.

Let us consider the case of $r \geqslant \sqrt{R}$, and assume that $x_1 \geqslant 2R_0$. Then

$$|W(x-y)| \leqslant CR^{-\delta} [s(x-y^\circ)+1]^{-\varepsilon}, \quad \forall y \subset D_0$$

where $y^\circ = (0, y_2, y_3)$. Setting

$$\varphi_0(y_2, y_3) = \int\limits_{-R_0}^{R_0} |f(y_1, y_2, y_3)|\, dy_1$$

we obtain

$$|I_{12}| \leqslant CR^{1-\delta} \quad \Delta_{1,\varepsilon} \max_{d_2} |\varphi_0(y_2, y_3)| \qquad (2.5)$$

It can be ascertained that

$$[s(x-y^\circ)+1]^{-\varepsilon} \leqslant C\left(\frac{y_2^2+y_3^2}{R}+1\right)^{-\varepsilon}, \quad \forall y \in d_3$$

and, consequently,

$$|I_{13}(x)| \leqslant CR^{-\delta} \int\limits_{\theta r/4}^{R_0} \varphi_1(\rho)\rho\left(\frac{\rho^2}{R}+1\right)^{-\varepsilon} d\rho$$

$$\varphi_1(\rho) = \int\limits_0^{2\pi} \varphi_0(\rho\cos\varphi, \rho\sin\varphi)\, d\varphi$$

Since on these assumptions $r \geqslant \sqrt{1/_2\, R(s+1)}$, hence, by setting $R_1 = \theta / (4\sqrt{2}) \sqrt{R(s+1)}$, we obtain

$$|I_{13}(x)| \leqslant CR^{-\delta}\int_{R_1}^{R_0}\rho\left(\frac{\rho^2}{R}+1\right)^{-\varepsilon}\varphi_1(\rho)\,d\rho \tag{2.6}$$

As regards $I_{11}(x)$, it is obvious that

$$|I_{11}(x)| \leqslant CR^{-\delta}[s(x)+1]^{-\varepsilon}\int_0^{2R_1}\rho\varphi_1(\rho)\,d\rho \tag{2.7}$$

It was previously assumed that $x_1 \geqslant 2R_0$. If $x_1 < 2R_0$, then, retaining estimates (2.6) and (2.7), we can show that

$$|I_1(x)| \leqslant J_{11}(x) + J_{13}(x) \tag{2.8}$$

where $J_{11}(x)$ and $J_{13}(x)$ are the right-hand sides of inequalities (2.7) and (2.6), respectively. It can be shown that the inequality (2.8) is, also, valid for $r < \sqrt{R}$.

2.4. Let us estimate $\varphi_1(\rho)$ and derive the inequalities for $|I_{1j}|$, where $j = 1, 2, 3$.

Propsition 1. Let $\beta+\gamma \leqslant 3$. If $\beta \geqslant 1+\gamma$, then

$$\varphi_1(\rho) \leqslant C(\rho+1)^{1-\beta-\gamma}\Delta_{1,\beta-\gamma}$$

if, however $\beta < 1+\gamma$, then for $\beta > 1$

$$\varphi_1(\rho) \leqslant C\begin{cases}(\rho+1)^{2-2\beta}, & \rho \leqslant \sqrt{R_0}\\ R_0^{1-\beta+\gamma}\rho^{-2\gamma}, & \rho > \sqrt{R_0}\end{cases}$$

Simple evaluations of the integral $\varphi_1(\rho)$ prove this proposition.

Using Proposition 1, we can reduce inequalities (2.5) – (2.7) for $\beta \geqslant 1+\gamma$ to the form

$$|I_{12}(x)| \leqslant CR^{\xi-\delta}(s+1)^{\xi-1}\Delta_{1,\beta-\gamma}\Delta_{1,\varepsilon} \tag{2.9}$$

$$|I_{13}(x)| \leqslant CR^{\xi-\delta}(s+1)^{\xi-\varepsilon}\Delta_{1,\beta-\gamma} \tag{2.10}$$

$$|I_{11}(x)| \leqslant CR^{\xi-\delta}(s+1)^{\xi-\varepsilon}\Delta_{3,\beta+\gamma}\Delta_{1,\beta-\gamma} \tag{2.11}$$

$$(\xi = {}^1/_2(3-\beta-\gamma))$$

For $1 < \beta < 1+\gamma$ we obtain

$$|I_{12}(x)| \leqslant CR^{2-\beta-\delta}(s+1)^{-\gamma}\Delta_{1,\varepsilon} \tag{2.12}$$

$$|I_{13}(x)| \leqslant CR^{2-\beta-\delta}(s+1)^{1-\gamma-\varepsilon} \tag{2.13}$$

$$|I_{11}(x)| \leqslant CR^{2-\beta-\delta}[1+(s+1)^{1-\gamma}\Delta_{1,\gamma}](s+1)^{-\varepsilon} \tag{2.14}$$

Summarizing estimates (2.2), (2.3) and (2.9) – (2.14), we come to the following proposition.

Proposition 2. Let $2-\beta+h_0 < \delta < 2+h_0$ and $\beta+\gamma \leqslant 3$. Then for $\beta \geqslant 1+\gamma$

$$|I(x)| \leqslant C\{R^{\xi-\delta}(s+1)^{\xi-1}[\Delta_{1,\beta-\gamma}+\Delta_{3,\beta+\gamma}]+R^{2-\beta-\delta+h_0}(s+1)^{-\tau_h}\Delta_{0,h}\}$$

and for $\beta < 1+\gamma$

$$|I(x)| \leqslant CR^{2-\beta-\delta}\{[(s+1)^{-\varepsilon}+(s+1)^{-\gamma}]\Delta_{1,\gamma}+R^{h_0}(s+1)^{-\tau_h}\Delta_{0,h}\Delta_{2,\delta}\}$$

where $\tau_h = \min(\omega_h, \zeta_h)$.

2.5. Let us determine the principal term of the velocity asymptotics. On the basis of formula (1.8) we have the following estimate for the surface integral $J_0(x)$:

$$| J_0(x) | \leqslant CR^{-1}(s+1)^{-1}$$

Assuming that estimate (1.6) is valid for $\alpha > 1/2$ and setting $\delta = \varepsilon = 3/2$, we apply to the volume integral J_d Proposition 2. Taking into consideration that $\beta = 2\alpha > 1$ and $\gamma = 0$, we obtain

$$| J_d(x) | \leqslant CR^{-\alpha} \quad (s+1)^{-(\alpha - 1/2)} \tag{2.15}$$

If $\alpha \leqslant 1$, these estimates yield

$$| v(x) | \leqslant CR^{-\alpha}(s+1)^{-(\alpha-1/2)}$$

which is a refinement of estimate (1.6). Note that the nonlinear terms in the integral J_d have been eliminated here. This was possible because condition $\alpha > 1/2$ ensures the definite "smallness" of v.

A further application to J_d of Proposition 2 on the assumption that $\beta = 2\alpha$, $\gamma = 2\alpha - 1 - h$ (h is a reasonably small positive number) yields

$$| v(x) - J_0(x) | \leqslant CR^{-\alpha_1} \quad (s+1)^{-\varkappa_1}$$
$$\alpha_1 = 2\alpha - (1+h)/2, \qquad \varkappa_1 = 2\alpha - 1 + h/2$$

Repetition of this reasoning yields an estimate similar to (2.15) but with exponents

$$\alpha_2 = 2\alpha_1 - \frac{1+h}{2}, \qquad \varkappa_2 = 2\alpha_1 - 1 + \frac{h}{2}$$

Thus, after a finite number of steps, we obtain

$$| v(x) | \leqslant CR^{-1}$$

The application of Proposition 2 yields the estimate

$$| v(x) - J_0(x) | \leqslant CR^{-1}(s+1)^{-1/2}$$

and after a further application of this proposition, we obtain

$$| v(x) - J_0(x) | \leqslant CR^{-(3-h)/2}(s+1)^{-(1-h)/2}$$

From this follows

$$| v(x) | \leqslant CR^{-1}(s+1)^{-1+h}$$

Setting $\beta = 2$ and $\gamma = 2 - 2h$ and applying again Proposition 2, we finally obtain

$$| J_d(x) | \leqslant CR^{-3/2}(s+1)^{-1/2} \log R \tag{2.16}$$

2.6. Let us determine the asymptotics of the surface integral $J_0(x)$. Expanding $H_{kl}(x-y)$, $\partial H_{kl}(x-y) / \partial y_j$ and $q_k(x-y)$ into series in powers of y and taking into account that for large R

$$\left| \frac{\partial^2 H_{kl}(x-y)}{\partial y_i \partial y_j} \right| \leqslant CR^{-2}(s+1)^{-2}$$

we obtain

$$J_{0k}(x) = \sum_{l=1}^{3} a_l H_{kl}(x) + \sum_{j,l=1}^{3} b_{jl} \frac{\partial H_{kl}(x)}{\partial x_j} + O[R^{-2}(s+1)^{-1}], \quad k = 1, 2, 3$$

where a_l and b_{lj} are certain constants. Individual terms in formula (2.17) are formally arranged so that terms which decrease inside the wake (*) (see footnote on page 658) as $R^{-3/2}$ are related to principal terms. In order to retain the symmetry of formulas no rearrangement has been carried out. We set

$$v_k^1(x) = \sum_{l=1}^{3} H_{kl}(x)\, a_l$$

and

$$v(x) = v^1(x) + w^{3/2}(x) \qquad (2.18)$$

It follows from (2.16) and (2.17) that

$$|w^{3/2}| \leqslant CR^{-3/2}(s+1)^{-1/2}\log R$$

Hence the decrease of $w^{3/2}$, which can be considered to be the error of the asymptotic formula, is of the order of $^3/_2$, which is reflected in its symbol.

3. Derivation of further terms of asymptotics. 3.1. Let us establish certain lemmas. First, let us consider integral (2.1), assuming, in addition to the assumptions made in Sect. 2, that function $W(x)$ is continuously differentiable for $x \neq 0$ and that

$$\left|\frac{\partial W(x)}{\partial x_1}\right| \leqslant CR^{-\delta-1}(s+1)^{-\varepsilon}$$
$$\left|\frac{\partial W(x)}{\partial x_k}\right| \leqslant CR^{-\delta-1/2}(s+1)^{-1/2-\varepsilon}, \qquad k = 2, 3 \qquad (3.1)$$

We shall further assume that the estimate of $f(x)$ contains a logarithmic multiplier, i.e.

$$|f(x)| \leqslant CR^{-\beta}(s+1)^{-\gamma}\log^{\beta_0} R$$

and that $\beta > 2$ and $3 < \beta + \gamma \leqslant 4$. Owing to the presence of the multiplier $\log^{\beta_0} R$ in this formula, the estimates of integrals I_2, I_3, I_{12} and I_{13} will, obviously, contain this multiplier.

Let us determine the asymptotics of $I(x)$ on the assumption that $r > \sqrt{R}$. We consider integral $I_{11}(x)$. Using (3.1), we obtain

$$\left|I_{11}(x) - W(x)\int_{d_1} f(y)\,dy\right| \leqslant CR^{-\delta-1/2}(s+1)^{-\varepsilon}\times$$
$$\int_{d_1} |f(y)|\left[\frac{|y_1|}{R^{1/2}} + \frac{|y_2|+|y_3|}{(s+1)^{1/2}}\right]dy$$

Let us evaluate the right-hand side of this inequality. It can be shown that

$$\int_{d_1} |f(y)|\,|y_1|\,dy \leqslant CR^{3-\beta}\log^{\beta_0} R\Delta_{2,\,\beta-\gamma} \qquad (\beta \leqslant \gamma + 2)$$

$$\int_{d_1} |f(y)|\,|y_1|\,dy \leqslant CR^{\xi+3/2}(s+1)^{\xi+1/2} \qquad (\beta > 2+\gamma)$$

It can be shown with the use of Proposition 1 that for $\beta \geqslant 1 + \gamma$

$$\int_{d_1} |f(y)|(|y_2|+|y_3|)\,dy \leqslant CR^{\xi+1/2}(s+1)^{\xi+1/2}\log^{\beta_0} R\Delta_{1,\,\beta-\gamma}\Delta_{4,\,\beta+\gamma}$$

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*) Region $\{x : x_1 > 0,\ x_2^2 + x_3^2 \leqslant cx_1\}$ is assumed here to define the wake.

which implies that

$$\left| I_{11}(x) - W(x) \int_{d_1} f(y)\, dy \right| \leqslant CR^{3/2-\delta} (s+1)^{-\varepsilon} \times [R^{1/2-\beta} + R^{\xi-3/2} (s+1)^{\xi}] \log^{\beta_0} R$$

In the derivation of the final formula we take into account that for $\beta \geqslant 1+\gamma$

$$\left| \int_{R^3 \setminus d_1} f(y)\, dy \right| \leqslant C\, [R^{2-\beta} + R^{\xi} (s+1)^{\xi}] \log^{\beta_0} R \Delta_{1,\, \beta-\gamma}$$

Hence

$$\left| I_{11}(x) - W(x) \int_{R^3} f(x)\, dx \right| \leqslant CR^{3/2-\delta} (s+1)^{-\varepsilon} [R^{1/2-\beta} + R^{\xi-3/2} (s+1)^{\xi}] \log^{\beta_0} R \tag{3.2}$$

The validity of inequality (3.2) for $r \leqslant \sqrt{R}$ can be readily proved.

Proposition 3. Let $\beta > 2, 3 < \beta + \gamma \leqslant 4$ and $\beta \geqslant 1 + \gamma$. Then

$$\left| I(x) - W(x) \int_{R^3} f(x)\, dx \right| \leqslant CR^{3/2-\delta} [R^{1/2-\beta} (s+1)^{-\varepsilon} + R^{\xi-3/2} (s+1)^{\xi-1}] \log^{\beta_0} R \Delta_{1,\beta-\gamma} \Delta_{4,\beta+\gamma} + CR^{2-\beta-\delta} \log^{\beta_0} R (s+1)^{-\tau_h} \Delta_{0,h} \tag{3.3}$$

Proof. The estimate of integrals I_2 and I_3, obtained above, is, evidently, valid in this case. For $x_1 \geqslant 2R_0$ integral I_1 was subdivided into three parts, and it was found that estimates (2.9), (2.10), (2.12) and (2.13) together with (3.2) yield (3.3). For $x_1 < 2R_0$, we subdivide I_1 into two components, viz. $I_1 = J_{11} + J_{13}$ in conformity with the subdivision of region D into g_1 and $g_3 = D \setminus g_1$, where

$$g_1 = \left\{ y : |y_1| \leqslant R_0,\ y_2^2 + y_3^2 \leqslant \left(\frac{\theta}{4}\right)^2 \frac{R(s+1)}{2} \right\}$$

Formula (3.2) and inequality (2.6) are then valid for J_{11} and J_{13}, respectively, and, consequently, inequalities (2.10) and (2.13) are also valid. This proves that inequality (3.3) is valid in this case.

3.2. Let us determine subsequent terms of velocity asymptotics. We denote $J_d(x)$ by $J_d(x; v, v)$, thus stressing that J_d is a quadratic functional of v. Using expansion (2.18), we obtain

$$J_d(x; v, v) = J_d(x; v^1, v^1) + J_d(x; v^1, w^{3/2}) + J_d(x, w^{3/2}, v^1) + J_d(x; w^{3/2}, w^{3/2})$$

The asymptotics of the last three terms in this formula can be determined on the basis of Proposition 3. Since the principal terms of that asymptotics are of the same form as the terms of order $R^{-3/2}$ in formula (2.17), hence only the constants b_{jl} are different in the asymptotic formula for $v(x)$. We denote these new constants by a_{jl}. Hence only the last term needs be evaluated. Using Proposition 3 and setting $\beta = 5/2$, $\gamma = 3/2$ and $\beta_0 = 1$, we find that in this case the residual term is $O\,[R^{-2}(s+1)^{-1} \log^3 R]$. Thus, setting

$$v_k^{3/2}(x) = \sum_{j,\,l=1}^{3} a_{jl}\frac{\partial H_{kl}}{\partial x_j} + J_{dk}(x;\ v^1,\ v^1), \quad k=1,2,3 \tag{3.4}$$

we obtain

$$v_k(x) = v_k^{(1)}(x) + v_k^{3/2}(x) + O\,[R^{-2}(s+1)^{-1/2}\log^3 R] \tag{3.5}$$

Note that here integrals $J_{dk}(x;\ v^1,\ v^1)$ are related to terms of order $R^{-3/2}$. The determination of asymptotics of these integrals is rather cumbersome. However for $a_2 = a_3 = 0$ it is readily done with the use of Proposition 2.

In fact, for $k \neq 1$

$$|\,H_{k1}(x)\,| \leqslant C\,[R^{-2} + R^{-3/2}(s+1)^{-1}]$$

and, consequently, for $a_2 = a_3 = 0$

$$|\,v_k^1(x)\,| \leqslant C\,[R^{-2} + R^{-3/2}(s+1)^{-1}], \quad k=2,3$$

We note that, with inequality (1.9) taken into account, J_{dk} is the sum of integrals to which either Proposition 2 (and then $\delta = 2$) or 3 is applicable. Hence in this case

$$v_k^{3/2}(x) = \sum_{j,\,l=1}^{3} a_{jl}\frac{\partial H_{kl}}{\partial x_j} \tag{3.6}$$

According to one of the results obtained in [11] $(a_1,\ a_2,\ a_3)$ is the vector of the force exerted by the fluid on the body. If $a_2 = a_3 = 0$, then the total force reduces to head drag. It can be shown that coefficients a_{jl} are expressed in terms of the resultant moment of forces exerted by the fluid on the body.

3.3. Let us determine the principal terms of the asymptotics of the velocity vector derivatives. These are obtained by formal differentiation of expression (2.18). Let $t = (t_1,\ t_2,\ t_3)$ be an arbitrarily small vector. It is obvious that

$$v(x+1) - v(x) = v^1(x+t) - v^1(x) + J_d(x+t) - J_d(x) + O\,[|\,t\,|\,R^{-2}(s+1)^{-1}]$$

The remainder $J_d(x+t) - J_d(x)$ can be estimated with the use of Proposition 2. We have

$$J_{dk}(x+t) - J_{dk}(x) = \int\limits_{|y-x|\leqslant 1}\sum_{j,\,l=1}^{3} v_j(y)\,v_l(y)\left[\frac{\partial H_{kl}(x+t-y)}{\partial y_j} - \frac{\partial H_{kl}(x-y)}{\partial y_j}\right]dy + \int\limits_{|y-x|\geqslant 1}\sum_{j,\,l=1}^{3} v_j(y)\,v_l(y)\,H_{klj}(x-y,\ t)\,dy$$

Applying Proposition 2 to the second integral and, since

$$|\,H_{klj}|\,(x-y,\ t)\,| \leqslant C\,|\,t\,|\,\,x-y\,|^{-2}(s+1)^{-2}$$

hence, by setting $\beta = \gamma = \beta_0 = 2$, we find that the considered integral is $O\,[|\,t\,|\,R^{-2}(s+1)^{-1}\log^4 R]$. For the first integral we have the inequality

$$\left|\int\limits_{|y-x|\leqslant 1}\sum_{j,\,l=1}^{3} v_j(y)\,v_l(y)\left[\frac{\partial H_{kl}(x+t-y)}{\partial y_j} - \frac{\partial H_{kl}(x-y)}{\partial y_j}\right]dy \leqslant CR^{-2}(s+1)^{-1}|\,t\,|\log\frac{1}{|\,t\,|}\right.$$

Thus

$$v(x+t)-v(x)=\sum_{j=1}^{3} t_j \frac{\partial v^1(x)}{\partial x_j}+|t|\log\frac{1}{|t|}O[R^{-2}(s+1)^{-1}\log^4 R] \quad (3.7)$$

We differentiate expression (1.13), noting that $J_d(x)$ can be differentiated according to the rules of differentiation of integrals with a weak singularity [12]. Thus

$$\frac{\partial J_{dk}(x)}{\partial x_i}=\sum_{j,l=1}^{3}\alpha_{kijl}v_j(x)v_l(x)-2\lambda\int_G\sum_{j,l=1}^{3}\frac{\partial^2 H_{kl}}{\partial x_i\partial y_j}v_jv_l dy$$

where the volume integral is singular. Note that Proposition 2 is applicable to the integral

$$\int_{G_x}\frac{\partial^2 H_{kl}}{\partial x_i\partial y_j}v_jv_l dy, \qquad G_x=\dot{G}\setminus\{y:|y-x|\leqslant 1\}$$

Furthermore

$$\int_{|y-x|\leqslant 1}\sum_{j,l=1}^{3}\frac{\partial^2 H_{kl}(x-y)}{\partial x_i\partial y_j}v_j(y)v_l(y)dy=\int_{|y-x|\leqslant 1}\frac{\partial^2 H_{kl}(x-y)}{\partial x_i\partial y_j}\times$$

$$[v_j(y)v_l(y)-v_j(x)v_l(x)]dy+\sum_{j,l=1}^{3}v_j(x)v_l(x)\times$$

$$\int_{|y-x|\leqslant 1}\frac{\partial^2 H_{kl}(x-y)}{\partial x_i\partial y_j}dy$$

By virtue of (3.7) the first term is $O[R^{-5/2}(s+1)^{-1}]$ and the second $O[R^{-2}(s+1)^{-1}]$. Thus

$$\frac{\partial v}{\partial x_j}=\frac{\partial v^1}{\partial x_j}+O[R^{-2}(s+1)^{-1}\log^4 R] \quad (3.8)$$

4. Asymptotics of the velocity vortex. **4.1.** It is possible by starting from formula (3.8) to determine the principal terms of asymptotics of $\omega=\operatorname{rot} v$. An elementary calculation yields

$$\omega=\frac{\lambda}{4\pi}\nabla s\times a\frac{e^{-\lambda s}}{R}+O[R^{-2}(s+1)^{-1}\log^4 R] \quad (4.1)$$

A refinement of the last term of this formula is presented below. Let us set

$$H_0(x-y)=\frac{e^{-\lambda s(x-y)}}{4\pi|x-y|} \quad (4.2)$$

Then

$$\omega_i(x)=\frac{\lambda}{2\pi}\int_G\sum_{j=1}^{3}(v_j\omega_i-v_i\omega_j)\frac{\partial}{\partial y_j}H_0(x-y)dy+\Omega_i(x) \quad (4.3)$$

$$\Omega_i(x)=\int_S\Big[H_0\frac{\partial\omega_i}{\partial n}-\omega_i\frac{\partial H_0}{\partial n}-2\lambda n_1 H_0\omega_i\Big]d\sigma-$$

$$\frac{\lambda}{2\pi}\int H_0(x-y)\sum_{j=1}^{3}(v_j\omega_i-v_i\omega_j)n_j d\sigma$$

Using formula (4.3), we derive the estimate of function $\varphi(x)=|\omega_1(x)|+|\omega_2(x)|+|\omega_3(x)|$ for large R. Formula (3.5) implies that

$$|v(x)|\leqslant CR^{-1}(s+1)^{-1}$$

since $|v^{3/2}(x)|\leqslant C_1R^{-1}(s+1)^{-1}$. The last inequality follows from Proposition 2 for $\beta = \gamma = 2$, and h is a small negative number. After some transformations we obtain

$$\sum_{i=1}^{3} |\omega_i(x) - \Omega_i(x)| \leqslant A \int_G \frac{\omega(y)\, Q_0(x-y)}{|y|\,[s(y)+1]} dy$$

$$Q_0(x) = R^{-3/2} \exp[-\lambda s(x)]\, [s^{1/2}(x) + R^{-1/2}]$$

Let us determine the asymptotics of integrals Ω_i inside the wake. Expanding $H_0(x - y)$ into series in powers of y, we obtain

$$\Omega_i(x) = \frac{e^{-\lambda s}}{R}\left(A_i + \sum_{j=2}^{3} B_{ij}\frac{x_j}{R}\right) + O(R^{-2}e^{-\lambda s})$$

A comparison of these expressions with (4.1) yields $A_i = 0$, $i = 1, 2, 3$, from which follows that

$$\sum_{i=1}^{3} |\Omega_i(x)| \leqslant AR^{-3/2}e^{-\mu s}$$

where μ is any number smaller than λ. Consequently

$$\varphi(x) \leqslant A \int_G \frac{\varphi(y)\, Q(x-y)}{|y|\,[s(y)+1]} dy + A\, \frac{e^{-\mu s(s)}}{R^{3/2}} \tag{4.4}$$

$$Q(x) = R^{-3/2} e^{-\mu s}(1 + R^{-1/2})$$

4.2. It follows from (4.1) that

$$|\omega(x)| \leqslant C_0 R^{-3/2}(s+1)^{-1} \tag{4.5}$$

Let us show that using inequality (4.4) the estimate (4.5) can be substantially refined, and that an exponential decrease of vorticity outside the wake can be obtained. The method developed in [10] in the course of analysis of the plane problem is also suitable in the case of three-dimensional space. This method is applied below.

Let us set $\mu = 2\mu_1 + \mu_2$, $\mu_1 > 0$ and $\mu_2 > 0$, and assume that the inequality

$$\varphi(y) \leqslant C_0 B^{l-1} [\mu_1 s(y) + 1]^{-l}\, \Gamma(l+1)\, |y|^{-3/2}, \quad \forall\, y \in G \tag{4.6}$$

is satisfied for $l = 1, 2, \ldots, n$. We shall show that this inequality is also valid for $l = n+1$.

Let us estimate the product $\psi(y)\exp[-\mu s(x-y)]$, using this assumption and setting $\psi(y) = |y|^{3/2}\varphi(y)$. We expand G into the sum of nonintersecting regions G_k, $k = 0, 1, \ldots, n$. Setting $s(x) = \tau$, we define these regions as follows:

$$G_k = \left\{y : \tau\left(\frac{1}{2} + \frac{k-1}{2n}\right) \leqslant s(y) < \tau\left(\frac{1}{2} + \frac{k}{2n}\right)\right\} \cap G, \quad k = 1, 2, \ldots, n-1$$

$$G_0 = \left\{y : s(y) < \frac{\tau}{2}\right\} \cap G, \qquad G_n = \left\{y : \tau\frac{2n-1}{2n} \leqslant s(y)\right\} \cap G$$

It can be shown that $s(x) - s(y) \leqslant s(x-y)$. Hence for $y \in G_k$

$$\exp[-\mu s(x-y)] \leqslant \exp[-\mu_2 s(x-y)] \exp\left[-\mu_1\tau\left(1 - \frac{k}{n}\right)\right]$$

from which, using the obvious inequality

$$\frac{t^m}{\Gamma(m+1)} < e^t, \qquad \forall t > 0, \quad m > 0$$

and the inductive assumption (4.6), for $k = 0, 1, \ldots n-1$ we obtain $\forall\, y \in G_k$

$$\psi(y) \exp[-\mu s(x-y)] \leqslant C \exp[-\mu_2 s(x-y)]\, eB^{k-1}\Gamma(k+1) \times$$
$$\Gamma(n-k+1)\left[\mu_1\tau\left(\frac{1}{2}+\frac{k-1}{2n}\right)+1\right]^{-k}\left[\mu_1\tau\left(1-\frac{k}{n}\right)+1\right]^{k-n}$$

It can be shown that

$$\left[\mu_1\tau\left(\frac{1}{2}+\frac{k-1}{2n}\right)+1\right]^{k}\left[\mu_1\tau\left(1-\frac{k}{n}\right)+1\right]^{n-k} \geqslant$$
$$(\mu_1\tau+1)^n\left(\frac{1}{2}+\frac{k-1}{2n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}$$

Using this inequality and the known inequalities

$$\sqrt{2\pi}\, m^{m+1/2}e^{-m} < \Gamma(m+1) < m^{m+1/2}e^{-m}\sqrt{2\pi}e, \qquad \forall m \geqslant 1$$

we obtain

$$\Gamma(k+1)\Gamma(n-k+1)\left[\mu_1\tau\left(\frac{1}{2}+\frac{k-1}{2n}\right)+1\right]^{-k}\left[\mu_1\tau\left(1-\frac{k}{n}\right)+1\right]^{k-n} \leqslant$$
$$\frac{2\pi e^{1-n}n^n}{(\mu_1\tau+1)^n}\left(\frac{2k}{n+k-1}\right)^k k^{1/2}(n-k)^{1/2} < \left(\frac{\pi}{2}e^3\right)^{1/2}\frac{\Gamma(n+3/2)}{(\mu_1\tau+1)^n}$$

Thus

$$\psi(y)\exp[-\mu s(x-y)] \leqslant \exp[-\mu_2 s(x-y)]\, C_0 B^{n-1} \times$$
$$\left(\frac{\pi}{2}e^5\right)^{1/2}\frac{\Gamma(n+3/2)}{(\mu_1\tau+1)^n}, \qquad \forall y \in \bigcup_{k=0}^{n-1} G_k$$

If $y \in G_n$, then

$$\psi(y)\exp[-\mu s(x-y)] < C_0 e B^{n-1}\frac{\Gamma(n+1)}{(\mu_1\tau+1)^n}$$

The last two inequalities imply that

$$\varphi(x) \leqslant C\left(\frac{\pi e^5}{2}\right)^{1/2} AB^{n-1}\frac{\Gamma(n+3/2)}{(\mu_1\tau+1)^n}\int_G |y|^{-3/2}[s(y)+1]^{-1} \times$$
$$Q_2(x-y)\,dy + A\frac{e^{-\mu_s(x)}}{R^{3/2}}$$
$$Q_2(x) = R^{-3/2}e^{-\mu_2 s(x)}(1+R^{-1/2})$$

Using Proposition 3, we obtain

$$\varphi(x) \leqslant C_0C_1\left(\frac{\pi e^5}{2}\right)^{1/2} AB^{n-1}\frac{\Gamma(n+3/2)}{(\mu_1\tau+1)^n}R^{-3/2}[s(x)+1]^{-3/2} + A\frac{e^{-\mu s(x)}}{R^{3/2}}$$

Let us select B so that

$$B \geqslant 2C_1A\left(\frac{\pi e^5}{2}\right)^{1/2}\mu_1\frac{\Gamma(n+3/2)}{\Gamma(n+2)}, \qquad n = 2, 3, \ldots \tag{4.7}$$

Then

$$\varphi(x) \leqslant \frac{C_0B^n}{2}\frac{\Gamma(n+2)R^{-3/2}}{[\mu_1 s(x)+1]^{n+1}} + \frac{Ae\Gamma(n+2)R^{-3/2}}{[\mu_1 s(x)+1]^{n+1}}$$

It can be assumed that $Ae \leqslant 1/2 B^n$, hence

$$\varphi(x) \leqslant C_0\frac{B^n\Gamma(n+2)}{[\mu_1 s(x)+1]^{n+1}}R^{-3/2}$$

It follows from (4.5) that the inequality (4.6) is satisfied for $n = 1$, hence it is satisfied for all n and, consequently,

$$\varphi(x) \leqslant C_0 \inf_n \frac{B^{n-1} n!}{[\mu_1 s(x) + 1]^n} R^{-3/2}$$

from which

$$\varphi(x) \leqslant C_1 R^{-3/2} \exp\left[-\frac{\mu_1}{B} s(x)\right] [s(x) + 1]^{1/2} \tag{4.8}$$

4.3. Let us show that

$$\varphi(x) < C_h e^{-(\lambda-h)s(x)} R^{-3/2}$$

where h is an arbitrarily small magnitude.

Let us consider the set M of such $\mu > 0$ for which

$$\sup_x R^{3/2} \varphi(x) \exp[\mu s(x)] < \infty \tag{4.9}$$

As previously proved, M is not an empty set. It is evident that, when $\mu_0 \in M$, then $[0, \mu_0] \subset M$. Hence M is either the interval $[0, m]$ or the half-open interval $[0, m)$. Let us assume that $m < \lambda$, and take $m_1 < m$, such that the remainder $m - m_1$ is fairly small. We set $\varphi_0(x) = \varphi(x)\, e^{m_1 s(x)}$. By definition

$$\varphi_0(x) \leqslant C_0 R^{-3/2} [s(x) + 1]^{-1} \tag{4.10}$$

From inequality (4.4) we obtain

$$\varphi_0(x) \leqslant A \int_G \frac{\varphi_0(y)\, Q_1(x-y)\, dy}{|y|\, [s(y) + 1]} + A \frac{e^{-\mu_1 s(x)}}{R^{3/2}} \tag{4.11}$$

$$\mu_1 = \mu - m_1, \quad Q_1(x) = R^{-3/2} e^{\mu_1 s(x)} (1 + R^{-1/2})$$

Note that, since the difference between μ and λ is as small as desired, $\mu - m_1 > 0$. Let us apply the results derived in Sect. 4.2 to function $\varphi_0(x)$. From inequality (4.11) follows that

$$\varphi_0(x) \leqslant C R^{-3/2} \exp\left[-\frac{\mu_3}{B} s(x)\right] [s(x) + 1]^{1/2} \tag{4.12}$$

where $\mu_3 = 1/3\, \mu_1$ can be assumed, and constant B is defined by inequalities (4.9) and independent of constant C_0 appearing in inequality (4.10). Hence, if m_1 is made reasonably close to m, we obtain $m_1 + \mu_3 / B > m$. Consequently, it follows from (4.12) that (4.9) is satisfied for $\mu < m_1 + \mu_3 / B$, which is absurd. This proves that $m = \lambda$ and, consequently,

$$|\omega(x)| \leqslant C_h R^{-3/2} \exp[-(\lambda - h)(R - x_1)] \tag{4.13}$$

4.4. Formula (4.1) can be readily refined. This is achieved in a manner similar to the proof of Proposition 2 and 3. Using inequality (4.13) after simple computations, we obtain

$$\omega = \frac{\lambda}{4\pi} \nabla s \times a \frac{e^{-\lambda s}}{R} + O\left[R^{-2} e^{-(\lambda-h) s(x)}\right]$$

REFERENCES

1. Leray, J., Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. Pure et Appl. Vol. 9, 1933.
2. Leray, J., Les problèmes non linéaires. Enseignement Math., Vol. 35, 1936.

3. Ladyzhenskaia, O. A., Investigation of the Navier-Stokes equation in the case of steady motion of incompressible fluid. Uspekhi Matem. Nauk, Vol. 4, №3, 1959.
4. Ladyzhenskaia, O. A., Mathematical Problems of Viscous Incompressible Fluid Dynamics. "Nauka", Moscow, 1970.
5. Finn, R., On the steady-state solutions of the Navier-Stokes equations III. Acta Math., Vol. 105, 1961.
6. Fujita, H., On the existence and regularity of the steady-state solutions of the Navier-Stokes equations. J. Fac. Sci., Univ. Tokyo, Ser. 1, Vol. 9, 1961.
7. Finn, R., On the steady-state solutions of the Navier-Stokes partial differential equations. Arch. Rat. Mech. and Analysis, Vol. 3, №5, 1959.
8. Clark, D., The vorticity at infinity for solutions of the stationary Navier-Stokes equations in external domains. Indiana Math. J., Vol. 20, № 7, 1971.
9. Pukhnachev, V. V., Estimation of the vortex velocity decrease at a distance from a body of revolution in a stream of viscous incompressible fluid. Collection: Dynamic of Continuous Medium, №8, Novosibirsk, 1971.
10. Babenko, K. I., The asymptotic behavior of a vortex far away from the body in a plane flow of viscous fluid. PMM Vol. 34, №5, 1970.
11. Finn, R., Estimates at infinity for stationary solutions of the Navier-Stokes equations. Bull. Math. de la Soc. Sci. Math. et Phys. de la RPR, Vol. 3(51), №4, 1959.
12. Mikhlin, S. G., Multiple Singular Integrals and Integral Equations, Fizmatgiz, Moscow, 1962.

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ON THE COMPLETENESS OF A SYSTEM OF ELEMENTARY SOLUTIONS OF THE BIHARMONIC EQUATION IN A SEMI-STRIP

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The problem of completeness of a system of elementary solutions in the space of biharmonic functions with finite energy is investigated. The problem arises during the study of infinite systems of linear algebraic equations in the asymptotic theory of plates. Actually a more general theory is developed here, including e.g. orthotropic and transversely inhomogeneous plates. The problem of existence of elementary solutions is solved at the same time. The results concerning the completeness obtained here are independent of the form of the boundary conditions at the end and can, consequently, be applied to a fairly wide class of elliptic boundary value problems which, in particular, appear in the theory of thick plates.

Before the problems of completeness are discussed, we study the problem of traces for the solution of a certain elliptic equation in a semi-cylinder. The necessary and sufficient conditions are formulated for the boundary values which